Towards Stable P2P-Streaming-Topologies

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1 Preliminaries

A *rooted tree* $T = (V, s; E)$ is a connected, acyclic (undirected) graph $(V + s, E)$, with a distinguished vertex s, called *root*. The root defines an orientation on the edges of T, such that every edge points away from s. A *successor* of a vertex v is another vertex $u \neq v$, such that there exists a path from v to u in the directed tree. A *predecessor* of v is another vertex $u \neq v$, such that u lies on the unique path from s to v.

The *depth* $\mathrm{depth}(v)$ of a vertex v in a rooted tree T is its distance from the root s. The *i-th level* $L_i(T)$ of T is the set of all vertices of depth i.

2 P2P-Streaming-Topologies and their Stability

Let s be a *server* in a computer network and $V = \{v_1, \ldots, v_n\}$ a set of n *clients*. Assume, that the server wants to distribute a multimedia stream to all n clients. The stream is split into k *stripes*, which may be routed independently. In a traditional client-server-setting the server has to send one copy of each stripe to every client. But in a Peer-to-Peer network, the clients can replicate the content and redistribute it to other clients. This framework allows the usage of more efficient and stable routings, than the classical hierarchical one.

In general each stripe is distributed among all clients along a tree whose root is the server. The combination of the trees for all k stripes, leads to *P2P-Streaming Topologies*.

Definition 2.1. *For $k \in \mathbb{N}$ a* P2P-streaming topology*, is a sequence $\mathcal{T} = (T_1, \ldots, T_k)$ of k directed trees $T_1 = (V + s, E_1), \ldots, T_k = (V + s, E_k)$ rooted at s.*

In this formal description, each tree T_i represents the routing of the i-th stripe, while each edge represents one packet, containing a copy of this stripe, transmitted between two clients.

The *degree* $\deg^{\mathcal{T}}(v)$ of a client or the server in an topology $\mathcal{T}$, is the sum of the degrees of v over all trees, ie.

$$\deg^{\mathcal{T}}(v) := \sum_{i=1}^{k} \deg^{T_i}(v).$$

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depth($\mathcal{T}$) is the maximal of all of trees in the topology $\mathcal{T}$, ie.

$$\text{depth}(\mathcal{T}) := \max\{\text{depth}(T_i) \mid 1 \le i \le k\}.$$

For a client v in $\mathcal{T}$, depth(v) is the maximal depth of v in $\mathcal{T}$, ie.

$$\text{depth}(v) := \max\left\{\text{depth}^{T_i}(v) \mid 1 \le i \le k\right\}.$$

For each tree T_i and each srt X of clients, we define $\text{succ}_i(X)$ as the set of all successors of clients in X. Furthermore we define a function $a_i^{\mathcal{T}} : \mathcal{P}(\mathcal{V}) \to \mathbb{N}$ with

$$a_i^{\mathcal{T}}(X) := |\text{succ}_i(X) \cup X|,$$

ie. $a_i^{\mathcal{T}}(X)$ is the number of all successors of elements in X, including X itself. Hence $a_i^{\mathcal{T}}$ counts the number of missing copies of stripe i, if the set X fails.

To obtain the total number of missing stripes, we have to sum over all trees, ie. $a^{\mathcal{T}} : \mathcal{P}(V) \to \mathbb{N}$ is given by $a^{\mathcal{T}}(X) := \sum_{i=1}^{k} a_i^{\mathcal{T}}(X)$.

Hence, $a^{\mathcal{T}}(X)$ measures the *total loss* of stripes, if the set X of clients fails. This includes the loss at the failed vertices. $a^{\mathcal{T}}$ measures the loss of stripes in the situation of an attack, where the vertices are forced to fail, and therefore the packets they lose have to be counted as loss.

In the situation of a failure of vertices, it does not seem to be apropriate, to count the packets which are sent to the failed clients. Instead, only the stripes not arriving at still operational clients are counted. This leads to $f^{\mathcal{T}} : \mathcal{P}(V) \to \mathbb{Z}$ with $f^{\mathcal{T}}(X) := a^{\mathcal{T}}(X) - k|X|$.

Definition 2.2. *Let $\mathcal{T}$ be a P2P-streaming-topology. Its* r-attack-stability $A^{\mathcal{T}}(r)$ *for $r \le nk$, is defined as*

$$A^{\mathcal{T}}(r) := \min\left\{|X| \mid X \subseteq V \text{ and } a^{\mathcal{T}}(X) \ge r\right\},$$

and its r-failure-stability $F^{\mathcal{T}}(r)$ *by*

$$F^{\mathcal{T}}(r) := \min\left\{|X| \mid X \subseteq V \text{ and } f^{\mathcal{T}}(X) \ge r\right\}.$$

In both cases we set $A^{\mathcal{T}}(r), F^{\mathcal{T}}(r) = \infty$, if no set exists with $a^{\mathcal{T}}(X), f^{\mathcal{T}}(X) \ge r$.

Our aim is to describe optimal stable topologies in certain classes of P2P-streaming-topologies. In general these may vary with r. But in the cases we examine, we will see that there exist topologies in the following universal sense.

Definition 2.3. *We say $\mathcal{T}$* is at least as stable as *$\mathcal{S}$, or $\mathcal{T} \succeq \mathcal{S}$, if $A^{\mathcal{T}}(r) \ge A^{\mathcal{S}}(r)$ for all r. $\mathcal{T}$ is said to be* more stable *than $\mathcal{S}$, or $\mathcal{T} \succ \mathcal{S}$, if $\mathcal{T} \succeq \mathcal{S}$ and there exists at least one r with $A^{\mathcal{T}}(r) > A^{\mathcal{S}}(r)$.*

Let $\mathfrak{C}$ be a class of P2P-streaming-topologies. A topology $\mathcal{T}$ is optimal stable in *$\mathfrak{C}$, if $\mathcal{T} \succeq \mathcal{S}$ for all $\mathcal{S} \in \mathfrak{C}$.*

2.1 Attack- versus Failure-Stability

In Definition 2.3 we only considered attack-stability. This restriction is justified by the following results.

Lemma 2.4. *For each P2P-streaming-topology $\mathcal{T}$ and $0 \le r \le nk$ we have*

$$F^{\mathcal{T}}(r) = A^{\mathcal{T}}(r + kF^{\mathcal{T}}(r)) \quad \text{and} \quad A^{\mathcal{T}}(r) = F^{\mathcal{T}}(r - kA^{\mathcal{T}}(r)).$$

Proof. Let $X \subseteq V$ with $|X| = F^{\mathcal{T}}(r)$ and $f^{\mathcal{T}}(X) \ge r$. Then

$$a^{\mathcal{T}}(X) = f^{\mathcal{T}}(X) + k|X| = f^{\mathcal{T}}(X) + kF^{\mathcal{T}}(r) \ge r + kF^{\mathcal{T}}(r).$$

Now assume, there exists a subset $Y \subseteq V$ with $|Y| < |X|$ and $a^{\mathcal{T}}(Y) \ge r + kF^{\mathcal{T}}(r)$. Then

$$f^{\mathcal{T}}(Y) = a^{\mathcal{T}}(Y) - k|Y| \ge r + kF^{\mathcal{T}}(r) - k|Y| = r + k(|X| - |Y|) > r.$$

Hence $|X| > F^{\mathcal{T}}(r)$, contradicting our assumptions.

The equality $A^{\mathcal{T}}(r) = F^{\mathcal{T}}(r - kA^{\mathcal{T}}(r))$ follows similar. □

Lemma 2.5. *Let $\mathfrak{C}$ be a class of P2P-streaming-topologies. $\mathcal{T} \in \mathfrak{C}$ is optimal stable in $\mathfrak{C}$ if and only if $F^{\mathcal{T}}(r) \ge F^{\mathcal{S}}(r)$ for every $\mathcal{S} \in \mathfrak{C}$ and $1 \le r \le nk$.*

Proof. If $\mathcal{T}$ is optimal stable in $\mathfrak{C}$, then we have $A^{\mathcal{T}}(r) \ge A^{\mathcal{S}}(r)$ for every $\mathcal{S} \in \mathcal{C}$ and $0 \le r \le nk$. Hence, by Lemma 2.4,

$$\begin{aligned} F^{\mathcal{T}}(r) &= A^{\mathcal{T}}(r + kF^{\mathcal{T}}(r)) \\ &\ge A^{\mathcal{S}}(r + kF^{\mathcal{T}}(r)) = F^{\mathcal{S}}\left(r + kF^{\mathcal{T}}(r) - kA^{\mathcal{S}}(r + kF^{\mathcal{T}}(r))\right). \end{aligned}$$

We have $A^{\mathcal{S}}(r + kF^{\mathcal{T}}(r)) \le A^{\mathcal{T}}(r + kF^{\mathcal{T}}(r))$, and therefore

$$\begin{aligned} &F^{\mathcal{S}}(r + kF^{\mathcal{T}}(r) - kA^{\mathcal{S}}(r + kF^{\mathcal{T}}(r))) \\ &\qquad \ge F^{\mathcal{S}}(r + kF^{\mathcal{T}}(r) - kA^{\mathcal{T}}(r + kF^{\mathcal{T}}(r))) = F^{\mathcal{S}}(r), \end{aligned}$$

because $F^{\mathcal{S}}(r)$ is monotone in r by definition.

The other direction follows similar. □

2.2 Stability Criterions

Lemma 2.6. *Let $\mathcal{T}, \mathcal{S}$ be two topologies, such that $a^{\mathcal{T}}(X) \ge a^{\mathcal{S}}(X)$ for all $X \subseteq V$. Then $\mathcal{S} \succeq \mathcal{T}$.*

Proof. For r let $X \subseteq V$ be a set, such that $|X| = a^{\mathcal{S}}(r)$ and $a^{\mathcal{S}}(X) \ge r$. Then

$$r \le a^{\mathcal{S}}(X) \le a^{\mathcal{T}}(X)$$

and hence $A^{\mathcal{T}}(r) \le |X| = A^{\mathcal{S}}(r)$. □

Lemma 2.7. *Let $\mathcal{T}$ and $\mathcal{S}$ be two topologies on n clients. If $v_1, \dots, v_n$ is an order on the clients of $\mathcal{T}$, such that*

$$A^{\mathcal{T}}(r) = \min\left\{i \mid a^{\mathcal{T}}(X_i) \ge r\right\} \quad \text{for } 0 \le r \le nk \text{ and } X_i := \{v_1, \dots, v_i\},$$

and $u_1, \dots, u_n$ is an order on the clients of $\mathcal{S}$, such that

$$a^{\mathcal{S}}(Y_i) \ge a^{\mathcal{T}}(X_i) \quad \text{for } 1 \le i \le n \text{ and } Y_i := \{u_1, \dots, u_i\},$$

then $\mathcal{T} \succeq \mathcal{S}$.

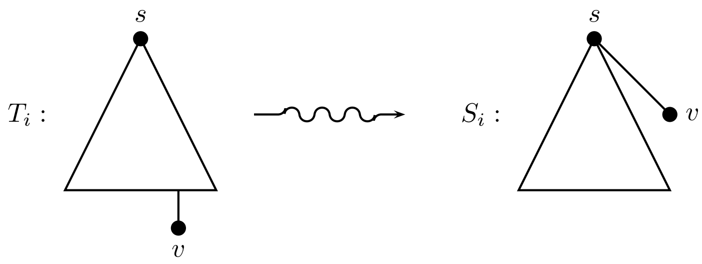

Figure 1: Increasing the degree of the server

Proof. For $0 \leq r \leq nk$ we have $A^{\mathcal{T}}(r) = \min\left\{i \mid a^{\mathcal{T}}(X_i) \geq r\right\}$. This implies

$$a^{\mathcal{S}}(X_{A^{\mathcal{T}}(r)}) \geq a^{\mathcal{T}}(X_{A^{\mathcal{T}}(r)}) \geq r,$$

and hence,

$$A^{\mathcal{S}}(r) \leq A^{\mathcal{T}}(r).$$

□

3 The Considered Topologies

In this paper we focus on two specific classes of topologies. Let $C, k \geq 1$ be natural numbers. The class $\mathfrak{C}^{C,k}$ contains all topologies with $n = Ck$ and $\deg^{\mathcal{T}}(s) \leq Ck$. The second class $\mathfrak{D}^{C,k}$ is the subclass of $\mathfrak{C}^{C,k}$, consisting of all topologies, such that exactly one stripe is sent from the server to each client.

Lemma 3.1. *Let $\mathcal{T} \in \mathfrak{C}^{C,k}$ be a topology, such that $\deg^{\mathcal{T}}(s) < Ck$. Then there exists a topology $\mathcal{S} \in \mathfrak{C}^{C,k}$, with $\deg^{\mathcal{T}}(s) < \deg^{\mathcal{S}}(s)$, such that $a^{\mathcal{T}}(X) \geq a^{\mathcal{S}}(X)$ for all $X \subseteq V$, and $a^{\mathcal{T}}(Y) > a^{\mathcal{S}}(Y)$ for at least one $Y \subseteq V$.*

Proof. Since $\deg^{\mathcal{T}}(s) < Ck$, there exists at least one tree T_i with $\text{depth}(T_i) \geq 2$, because otherwise every vertex would receive at least one stripe directly from the server. Set $\mathcal{S} := (T_1, \ldots, T_{i-1}, S_i.T_{i+1}, \ldots, T_k)$, where S_i is obtained from T_i, by moving one leaf v of T_i of depth 2 or higher, directly beneath the server (cmp. Figure 1). Now let $X \subseteq V$ be an arbitrary set of clients. If X contains neither a predecessor of v in T_i, nor v itself, then the set of successors of X in S_i coincides with that in T_i, and hence $a_i^{\mathcal{S}}(X) = a_i^{\mathcal{T}}(X)$. If X contains a predecessor of v in T_i but not v itself, then we have $\text{succ}^{S_i}(X) = \text{succ}^{T_i}(X) \setminus \{v\}$, leading to $a_i^{\mathcal{S}}(X) = a_i^{\mathcal{T}}(X) - 1$. If X contains v, but no predecessor of v, then $a_i^{\mathcal{S}}(X) = a_i^{\mathcal{T}}(X)$, since

$$a_i^{\mathcal{T}}(X) = a_i^{\mathcal{T}}(X - v) + a_i^{\mathcal{T}}(v) = a_i^{\mathcal{S}}(X - v) + a_i^{\mathcal{S}}(v).$$

If X contains v, as well as a predecesor of v in T_i, we have $\text{succ}^{T_i}(X) = \text{succ}^{S_i}(X)$, and hence $a_i^{\mathcal{T}}(X) = a_i^{\mathcal{S}}(X)$. Therefore $a_i^{\mathcal{S}}(X) \leq a_i^{\mathcal{T}}(X)$ for $X \subseteq V$. Since all other trees remain unchanged, we have

$$a^{\mathcal{S}}(X) \leq a^{\mathcal{T}}(X).$$

□

By repeated application of Lemma 3.1, we obtain the following corollary.

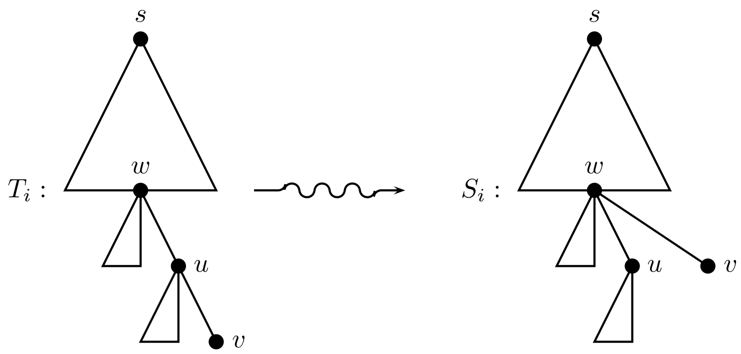

Figure 2: Reducing the depth of a topology

Corollary 3.2. *Let $\mathcal{T} \in \mathfrak{C}^{C,k}$ be a topology, such that $\deg^{\mathcal{T}}(s) < Ck$. Then there exists a topology $\mathcal{S} \in \mathfrak{C}^{C,k}$, with $\deg^{\mathcal{S}}(s) = Ck$, such that $\mathcal{S} \succeq \mathcal{T}$.*

Lemma 3.3. *Let $\mathcal{T} \in \mathfrak{C}^{C,k}$ be a topology with $\operatorname{depth}(\mathcal{T}) \geq 3$. Then there exists a topology $\mathcal{S} \in \mathfrak{C}^{C,k}$ with $\operatorname{depth}(\mathcal{S}) \leq 2$ and $a^{\mathcal{S}}(X) \leq a^{\mathcal{T}}(X)$ for all $X \subseteq V$ and $\deg^{\mathcal{S}}(s) = \deg^{\mathcal{T}}(s)$.*

Proof. Let T_i be a tree of maximum depth in $\mathcal{T}$. Then there exists a leaf v of depth at least 3 in T_i. Let u be its direct predecessor and w the direct predecessor of u (cmp. Figure 2). Since v has depth 3 or higher, we have $w \neq s$. Now construct S_i from T_i by moving v from its predecessor u to its predecessor w, ie. v is moved to the same depth as u. As a consequence, the number of successors of an arbitrary client in S_i is the same as in T_i, with the exception of u. In S_i u has one successor - v - less than it has in T_i. Hence, $a_i^{\mathcal{T}}(X) \geq a_i^{\mathcal{S}}(X)$ for every $X \subseteq V$. For $\mathcal{S} := (T_1, \ldots, T_{i-1}, S_i, T_{i+1}, \ldots, T_k)$, this implies $a^{\mathcal{T}}(X) \geq a^{\mathcal{S}}(X)$.

This construction reduces the number of clients of maximum depth by one. Repeated application leads to a topology $\mathcal{S}$ with maximum depth 2 and $a^{\mathcal{S}}(X) \leq a^{\mathcal{T}}(X)$ for every $X \subseteq V$. □

Corollary 3.4. *Let $\mathcal{T} \in \mathfrak{C}^{C,k}$ be a topology with $\operatorname{depth}(\mathcal{T}) \geq 3$. Then there exists a topology $\mathcal{S} \in \mathfrak{C}^{C,k}$ with $\operatorname{depth}(\mathcal{S}) \leq 2$ and $\mathcal{S} \succeq \mathcal{T}$ and $\deg^{\mathcal{S}}(s) = \deg^{\mathcal{T}}(s)$.*

4 Topologies of Depth 2

Due to Corollary 3.4 we can restrict ourselves to topologies in $\mathfrak{D}^{C,k}$ with maximum depth 2. For these, the functions $a^{\mathcal{T}}$ and $f^{\mathcal{T}}$ can be written in a very intuitive form. First observe, that the failure of a client v causes a failure of all stripes he sends to other clients, which did not fail themselves. Hence, we have

$$f^{\mathcal{T}}(X) = \sum_{i=1}^{k} \sum_{v \in X} |\mathrm{succ}_i(v) \setminus X|,$$

where $\mathrm{succ}_i(v)$ is the set of all successors of v in T_i.

Since every client receives exactly one stripe directly from the server, we have $\mathrm{succ}_i(v) \neq \emptyset$ for at most one i, namely the unique i, such that v has depth 1 in T_i. This leads to

$$f^{\mathcal{T}}(X) = \sum_{i=1}^{k} \sum_{v \in L_1(T_i) \cap X} |\mathrm{succ}_i(v) \setminus X|,$$

where $L_1(T_i)$ is the first level of T_i, ie. the set of clients of depth 1 in T_i. In addition, the sets $\mathrm{succ}_i(v)$ for $v \in L_1(T_i)$ are a partition of the set $C \setminus L_1(T_i)$. Hence, the sets $\mathrm{succ}_i(v) \setminus X$ for $v \in L_1(T_i)$ are a partition of $V \setminus (X \cup L_1(T_i))$.

A topology $\mathcal{T} \in \mathfrak{D}^{C,k}$ of depth 2 induces orderings on the trees and on the clients. First, order the trees of $\mathcal{T}$ ascending by the sizes of their first level. Hence, we have $\mathcal{T} = (T_1, \dots, T_k)$ with $|L_1(T_1)| \leq \cdots \leq |L_1(T_k)|$. Since every client is at level one in exactly one tree, this induces an ordered partition $(L_1(T_1), \dots, L_1(T_k))$ on V. In the following we always assume, that the trees of $\mathcal{T}$ are ordered in this way.

Inside one set $L_1(T_i)$ of this partition, the clients may be ordered descending, by the number of successors, not already in preceding partitions, ie. they are ordered descending by

$$|\mathrm{succ}_i(v) \setminus \bigcup_{j=1}^{i-1} L_1(T_j)|.$$

This results in a total ordering $v_1, \dots, v_n$, such that

- $L_1(T_i) = \{v_{h+1}, \dots, v_{h+|L_1(T_i)|}\}$ for $h = \sum_{j=1}^{i-1} c_j$ and $c_j := |L_1(T_j)|$,
- $c_1 \leq \cdots \leq c_k$, and
- $|\mathrm{succ}_i(v_{h+1}) \setminus X_h| \geq \cdots \geq |\mathrm{succ}_i(v_{h+c_i}) \setminus X_h|$, with $X_h = \{v_1, \dots, v_h\}$.

Orderings of this type are called *proper*.

Lemma 4.1. *Let $v_1, \dots, v_n$ be a proper ordering of the clients of $\mathcal{T} \in \mathfrak{D}^{C,k}$. Then*

$$f^{\mathcal{T}}(X_{h+l}) \geq i(n-h-l) + l(k-i-1)$$

for $0 \leq i \leq k$, $0 \leq l < c_{i+1}$ and $h = \sum_{j=1}^{i} c_j$.

Proof. We have

$$f^{\mathcal{T}}(X_{h+l}) = \sum_{j=1}^{k} \sum_{v \in L_1(T_j) \cap X_{h+l}} |\mathrm{succ}_j(v) \setminus X_{h+l}|.$$

Since $L_1(T_j) \cap X_{h+l} = L_1(T_j)$ for $j \leq i$ and $L_1(T_{i+1}) \cap X_{h+l} = \{v_{h+1}, \dots, v_{h+l}\}$ and $L_1(T_j) \cap X_{h+l} = \emptyset$ for $j > i+1$, we obtain

$$f^{\mathcal{T}}(X_{h+l}) = \sum_{j=1}^{i} \sum_{v \in L_1(T_j)} |\mathrm{succ}_j(v) \setminus X_{h+l}| + \sum_{j=1}^{l} |\mathrm{succ}_{i+1}(v_{h+j}) \setminus X_{h+l}|.$$

Since the sets $\mathrm{succ}_j(v) \setminus X_{h+l}$ with $v \in L_1(T_j)$ and $j \leq i$ are a partition of $V \setminus X_{h+l}$, we have

$$f^{\mathcal{T}}(X_{h+l}) = \sum_{j=1}^{i} |V \setminus X_{h+l}| + \sum_{j=1}^{l} |\mathrm{succ}_{i+1}(v_{h+j}) \setminus X_{h+l}|.$$

Obviously, we have $|V \setminus X_{h+1}| = n - h - l$, leading to

$$f^{\mathcal{T}}(X_{h+l}) = i(n-h-l) + \sum_{j=1}^{l} |\text{succ}_{i+1}(v_{h+j}) \setminus X_{h+l}|.$$

Furthermore, we have $\text{succ}_{i+1}(v_{h+j}) \setminus X_{h+l} = \text{succ}_{i+1}(v_{h+j}) \setminus X_h$ for $1 \leq j \leq l$, since the vertex v_{h+j} is at level 1 in T_{i+1}. This leads to

$$f^{\mathcal{T}}(X_{h+l}) = i(n-h-l) + \sum_{j=1}^{l} |\text{succ}_{i+1}(v_{h+j}) \setminus X_h|. \tag{1}$$

Since $v_1, \ldots, v_n$ is a proper ordering, the vertices $v_{h+1}, \ldots, v_{h+c_{i+1}}$ satisfy

$$|\text{succ}_{i+1}(v_{h+1}) \setminus X_h| \geq \cdots \geq |\text{succ}_{i+1}(v_{h+c_{i+1}}) \setminus X_h|.$$

Following Lemma A.1, this implies

$$\sum_{j=1}^{l} |\text{succ}_{i+1}(v_{h+j}) \setminus X_h|$$

$$\geq l \left\lfloor \frac{|V \setminus X_{h+c_{i+1}}|}{c_{i+1}} \right\rfloor = l \left\lfloor \frac{n-h-c_{i+1}}{c_{i+1}} \right\rfloor = l \left\lfloor \frac{\sum_{j=i+2}^{k} c_j}{c_{i+1}} \right\rfloor$$

Since $c_1 \leq \cdots \leq c_k$, we furthermore have $\sum_{j=i+2}^{k} c_j \geq (k-i-1)c_{i+1}$ and hence

$$\sum_{j=1}^{l} |\text{succ}_{i+1}(v_{h+j}) \setminus X_h| \geq l \left\lfloor \frac{\sum_{j=i+2}^{k} c_j}{c_{i+1}} \right\rfloor \geq l \left\lfloor \frac{(k-i-1)c_{i+1}}{c_{i+1}} \right\rfloor = l(k-i-1) \tag{2}$$

(1) in combination with (2) leads to

$$f^{\mathcal{T}}(X_{h+l}) \geq i(n-h-l) + l(k-i-1).$$

□

5 The optimal topology in $\mathfrak{D}^{C,k}$

We propose that, the following topology is optimal stable in $\mathfrak{D}^{C,k}$. Assume that we have an ordering $v_1, \ldots, v_n$ on the $n = Ck$ clients. Then the i-th tree S_i of the topology $\mathcal{C} \in \mathfrak{D}^{C,k}$ is constructed in the following way:

- The clients $v_{C(i-1)+1}, \ldots, v_{Ci}$ are directly connected to the server s.
- For $0 \leq j < k$ and $j \neq i-1$, the client v_{Cj+l} is connected to $v_{C(i-1)+l}$.

Observe, that the first level of every tree contains exactly C clients. Furthermore, client $v_{C(i-1)+l}$ has exactly $k-1$ successors in tree T_i, and none otherwise, and exactly $i-1$ of these successors are in the first level of a tree T_j with $j < i$. Hence $|\text{succ}_i(v_{C(i-1)+l})| = k-1$ and $|\text{succ}_i(v_{C(i-1)+l}) \setminus X_{C(i-1)}| = k-i$, where $X_i := \{v_1, \ldots, v_i\}$.

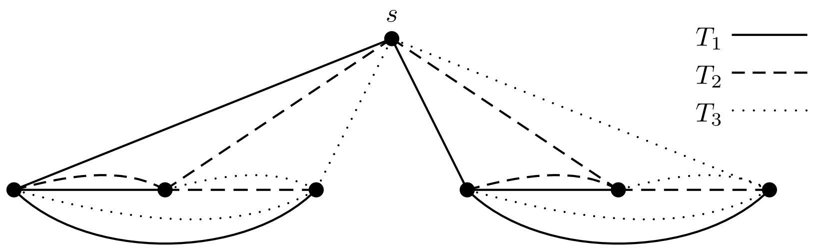

Figure 3: The stripes of the optimal topology $\mathcal{C}$ for $C = 2$ and $k = 3$

Alternatively, we can describe $\mathcal{C}$ in the following way. The clients are separated into groups $V_j := \{v_j, v_{C+j}, \ldots, v_{C(k-1)+j}\}$ for $1 \leq j \leq C$. The server sends a copy of the i-th stripe to $v_{C(i-1)+j}$. This client then distributes the stripe to all other members of its group. In total, the server sends $Ck = n$ packets to clients, and each client receives exaclty one stripe from the server, and sends $k-1$ stripes to peers.

Lemma 5.1. *For $0 \leq r \leq nk$, we have*

$$A^{\mathcal{C}}(r) = \min\{i \mid a^{\mathcal{C}}(X_i) \geq r\},$$

where $X_i := \{v_1, \ldots, v_i\}$.

Proof. First, observe, that the failure of a client in a specific group V_j, does not affect members of other groups. If l clients in a group fail, this leads to a total loss of $lk + l(k-l) = l(2k-l)$ stripes, because every failed client causes a loss of k incoming stripes and $k-l$ outgoing stripes to the remaining members.

Now, we take a look at the total loss caused by i failures. Assume, that l_j nodes failed in group $V_j j$, and hence $\sum_{j=1}^{C} l_j = i$. Then the total loss is

$$\sum_{j=1}^{C} kl_j + l_j(k-l_j) = 2ki - \sum_{j=1}^{C} l_j^2.$$

Hence, to maximize the total loss, we have to minimize $\sum_{j=1}^{C} l_j^2$ under the restriction $\sum_{j=1}^{C} l_j = i$.

Now assume that there exist two groups, eg. 1 and 2, such that $l_2 = l_1 + c$ for $c \geq 2$. Then we have

$$l_1^2 + l_2^2 = l_1^2 + (l_1+c)^2 = 2l_1^2 + 2cl_1 + c^2,$$

while

$$\begin{aligned}(l_1+1)^2 + &(l_2-1)^2 \\ &= l_1^2 + 2l_1 + 1 + l_1^2 + 2(c-1)l_1 + (c-1)^2 \\ &= 2l_1^2 + 2cl_1 + c^2 - 2(c-1).\end{aligned}$$

Since the second term is smaller than the first, it would be a better solution to increase l_1 and to decrease l_2 by one each. Repetiton of this procedure shows, that the maximal loss in $\mathcal{C}$ caused by the failure of i nodes, is realized by a distribution of these failures to the groups, such that the numbers of failures in two distinct groups differ by at most one.

Hence, the set $X_i = \{v_1, \dots, v_i\}$ realizes the maximal loss caused by the failure of i clients, and therefore $A^{\mathcal{C}}(r) = \min\{i \mid a^{\mathcal{C}}(X_i) \geq r\}$. □

Lemma 5.2. *For $0 \leq i \leq k$ and $1 \leq l < C$, we have*

$$f^{\mathcal{C}}(X_{Ci+l}) = i(n - Ci - l) + l(k - i - 1).$$

Proof. We have

$$f^{\mathcal{C}}(X_{Ci+l}) = \sum_{j=1}^{k} \sum_{v \in L_1(C_j) \cap X_{h+l}} |\mathrm{succ}_j(v) \setminus X_{Ci+l}|.$$

Since $L_1(C_j) \cap X_{Ci+l} = L_1(C_j)$ for $j \leq i$ and $L_1(C_{i+1}) \cap X_{Ci+l} = \{v_{Ci+1}, \dots, v_{Ci+l}\}$ and $L_1(C_j) \cap X_{Ci+l} = \emptyset$ for $j > i + 1$, we obtain

$$f^{\mathcal{C}}(X_{Ci+l}) = \sum_{j=1}^{i} \sum_{v \in L_1(C_j)} |\mathrm{succ}_j(v) \setminus X_{Ci+l}| + \sum_{j=1}^{l} |\mathrm{succ}_{i+1}(v_{Ci+j}) \setminus X_{Ci+l}|.$$

Since $\mathrm{succ}_j(v) \setminus X_{Ci+l}$ with $v \in L_1(C_j)$ and $j \leq \hat{i}$ is a partition of $V \setminus X_{Ci+l}$, we have

$$f^{\mathcal{C}}(X_{Ci+l}) = \sum_{j=1}^{i} |V \setminus X_{Ci+l}| + \sum_{j=1}^{l} |\mathrm{succ}_{i+1}(v_{Ci+j}) \setminus X_{Ci+l}|.$$

Obviously, we have $|V \setminus X_{Ci+1}| = n - Ci - l$, leading to

$$f^{\mathcal{T}}(X_{h+l}) = i(n - Ci - l) + \sum_{j=1}^{l} |\mathrm{succ}_{i+1}(v_{Ci+j}) \setminus X_{Ci+l}|.$$

Furthermore, we have $\mathrm{succ}_{i+1}(v_{Ci+j}) \setminus X_{Ci+l} = \mathrm{succ}_{i+1}(v_{Ci+j}) \setminus X_{Ci}$ for $1 \leq j \leq l$, since the vertex v_{Ci+j} is at level 1 in C_{i+1}. This leads to

$$f^{\mathcal{C}}(X_{Ci+l}) = i(n - Ci - l) + \sum_{j=1}^{l} |\mathrm{succ}_{i+1}(v_{Ci+j}) \setminus X_{Ci}|. \tag{3}$$

Due to its construction, $\mathrm{succ}_{i+1}(v_{Ci+j})$ contains $k - 1$ clients, with i of then in X_{Ci}. Hence,

$$f^{\mathcal{C}}(X_{Ci+l}) = i(n - Ci - l) + l(k - i - 1). \tag{4}$$

□

Lemma 5.3. *For $\mathcal{T} \in \mathfrak{D}^{C,k}$ and $1 \leq i \leq n$, we have $f^{\mathcal{T}}(X_i) \geq f^{\mathcal{C}}(X_i)$.*

Proof. We set

$$\hat{i} = \max\{h \mid \sum_{j=1}^{h} c_j \leq i\} \text{ and } \hat{l} = i - \sum_{j=1}^{\hat{i}} c_j.$$

With $h = \sum_{j=1}^{\hat{i}}$, this implies $i = h + l$ with $0 \leq \hat{i} \leq k$ and $0 \leq \hat{l} < c_{\hat{i}+1}$. Following Lemma 4.1, this implies

$$f^{\mathcal{T}}(X_i) \geq \hat{i}(n - h - \hat{l}) + \hat{l}(k - \hat{i} - 1) = \hat{i}(n - i) + \hat{l}(k - \hat{i} - 1). \tag{5}$$

Now set

$$\tilde{i} = \left\lfloor \frac{i}{C} \right\rfloor \text{ and } \tilde{l} = i - C\tilde{i}.$$

This implies $i = C\tilde{i} + \tilde{l}$ and $0 \leq \tilde{l} < C$ and by Lemma 5.2

$$f^{\mathcal{C}}(X_i) = \tilde{i}(n - C\tilde{i} - \tilde{l}) + \tilde{l}(k - \tilde{i} - 1) = \tilde{i}(n - i) + \tilde{l}(k - \tilde{i} - 1). \tag{6}$$

Due to Lemma A.1 and the fact that the c_i are non-decreasing, we have

$$\sum_{j=1}^{\hat{i}+1} c_j \leq (\hat{i} + 1) \left\lceil \frac{n}{k} \right\rceil = (\hat{i} + 1)C,$$

and as a consequence

$$\tilde{i} = \left\lfloor \frac{i}{C} \right\rfloor = \left\lfloor \frac{h + \hat{l}}{C} \right\rfloor < \left\lfloor \frac{h + c_{i+1}}{C} \right\rfloor \leq \left\lfloor \frac{(\hat{i} + 1)C}{C} \right\rfloor = \hat{i} + 1.$$

implying $\tilde{i} \leq \hat{i}$. Furthermore, observe

$$\tilde{l}(k - \tilde{i} - 1) < C(k - \tilde{i} - 1) = n - C(\tilde{i} + 1) < n - i.$$

If $\tilde{i} < \hat{i}$ we have

$$\begin{aligned} f^{\mathcal{C}}(X_i) = \tilde{i}(n - i) + \tilde{l}(k - \tilde{i} - 1) < \tilde{i}(n - i) + (n - i) = \\ = (\tilde{i} + 1)(n - i) \leq \hat{i}(n - i) \leq f^{\mathcal{T}}(X_i) \end{aligned} \tag{7}$$

If $\tilde{i} = \hat{i}$ we have

$$\tilde{l} = i - C\tilde{i} = i - C\hat{i} = h + l - C\hat{i} = \hat{l} + (h - C\hat{i}) \leq \hat{l},$$

since $h = \sum_{j=1}^{\hat{i}} c_j \leq C\hat{i}$. This implies

$$\begin{aligned} f^{\mathcal{T}}(X_i) \geq \hat{i}(n - i) + \hat{l}(k - \hat{i} - 1) = \\ = \tilde{i}(n - i) + \hat{l}(k - \tilde{i} - 1) \geq \tilde{i}(n - i) + \tilde{l}(k - \tilde{i} - 1)) = f^{\mathcal{C}}(X_i). \end{aligned} \tag{8}$$

□

Theorem 5.4. *$\mathcal{C} \succeq \mathcal{T}$ for every $\mathcal{T} \in \mathfrak{D}^{C,k}$.*

Proof. Since $a^{\mathcal{T}}(X) = f^{\mathcal{T}}(X) + k|X|$ for every $\mathcal{T} \in \mathfrak{D}^{C,k}$, the topologies $\mathcal{T}$ and $\mathcal{C}$ satisfy the conditions of Lemma 2.7, due to Lemma 5.3. Hence $\mathcal{S} \succeq \mathcal{T}$ for every $\mathcal{T} \in \mathfrak{D}^{C,k}$. □

6 Optimal topologies in $\mathfrak{C}^{C,k}$

Now, that we know a optimal stable topology of $\mathfrak{D}^{C,k}$, we examine the more general class $\mathfrak{C}^{C,k}$.

Lemma 6.1. *If $\mathcal{T} \in \mathfrak{C}^{C,k}$ is an optimal stable topology in $\mathfrak{C}^{C,k}$, then $a^{\mathcal{T}}(v) = 2k-1$ for every client v.*

Proof. First observe, $a^{\mathcal{C}}(v) = k + (k-1) = 2k-1$ for every client v, and hence, due to Lemma 5.1, $A^{\mathcal{C}}(2k-1) = 1$ and $A^{\mathcal{C}}(2k) > 1$. Since $\mathcal{C} \in \mathfrak{C}^{C,k}$, every optimal stable topology $\mathcal{T}$ in $\mathfrak{C}^{C,k}$ has to satisfy $A^{\mathcal{T}}(2k-1) \leq 1$. At the same time, since $2l - 1 \geq 0$, we have $A^{\mathcal{T}}(2k-1) \geq 1$, leading to

$$A^{\mathcal{T}}(2k-1) = 1.$$

Now assume, that there exists a client v with $a^{\mathcal{T}}(v) > 2k-1$. Then $A^{\mathcal{T}}(2k) = 1 < A^{\mathcal{C}}(2k)$, contradicting the optimality of $\mathcal{T}$. Hence we have $a^{\mathcal{T}}(v) \leq 2k-1$ for every client v.

Since $a^{\mathcal{T}}(v)$ counts the stripes lost due to the failure of v, it is at least the number of edges incident to v in all trees. Hence, summing up the $a^{\mathcal{T}}(v)$, counts every edge at least twice, except for those connected to the server. This results in

$$\sum_{v \in V} a^{\mathcal{T}}(v) \geq 2nk - \deg^{\mathcal{T}}(s) \geq 2nk - n = n(2k-1).$$

If $a^{\mathcal{T}}(v) < 2k-1$ for at least one client, we have

$$\sum_{v \in V} a^{\mathcal{T}}(v) < n(2k-1),$$

contradicting the preceding observation. Hence we have $a^{\mathcal{T}}(v) = 2k-1$ for every client v. □

Corollary 6.2. *If $\mathcal{T}$ is optimal stable in $\mathfrak{C}^{C,k}$, then $\mathcal{T} \in \mathfrak{D}^{C,k}$.*

Proof. Assume, that $\mathcal{T}$ has depth 3 or higher. Then in one tree t_i there exists a sequence $s \to u \to v \to w$. Hence, by summing up the values $a^{\mathcal{T}}(v)$, the edge $v \to w$ would be countet at least three times, once for w, once for v and once for u. This would lead to

$$\sum_{v \in V} a^{\mathcal{T}}(v) > 2nk - \deg^{\mathcal{T}}(s) \geq 2nk - n = n(2k-1),$$

and therefore $a^{\mathcal{T}}(c) > 2k-1$ for at least one client, contradicting Lemma 6.1. Hence, $\mathcal{T}$ has at most depth 2.

Now assume that one client receives more than one package. Then there exists another client v, which does not receive any stripe directly from the server, and hence has depth 2 in every tree. As a consequence v has no successor in $\mathcal{T}$, leading to $a^{\mathcal{T}}(v) = k$, contradicting Lemma 6.1.

As a consequence every client receives at most one stripe directly from the server. By Lemma 3.1, $\mathcal{T}$ has to satisfy $\deg^{\mathcal{T}}(s) = Ck$, because it is optimal stable in $\mathfrak{C}^{C,k}$, implying $\mathcal{T} \in \mathfrak{D}^{C,k}$. □

Lemma 6.3. *For every topology $\mathcal{T} \in \mathfrak{C}^{C,k}$ there exists a topology $\mathcal{S} \in \mathfrak{C}^{C,k}$ with $\mathcal{S} \succeq \mathcal{T}$.*

Proof. Let $\mathcal{T}$ be an arbitrary topology in $\mathfrak{C}^{C,k} \setminus \mathfrak{D}^{C,k}$, ie. there either exists a client receiving none, two or more stripes directly from the server. Folowing Lemma 3.1, we can assume that $\deg^{\mathcal{T}}(s) = Ck$ and $\text{depth}(\mathcal{T}) = 2$. Since there exists a client v receiving no stripe directly from s, there has to exist another client v, receiving at least two stripes directly from s. As a consequence, v is a leaf in every tree, while u is a tevel 1 in at least two trees. Assume that T_i is one of those trees. Then we define S_i by exchanging u and v, ie. u becomes a leaf in T_i and v is moved to level 1. This leads to a topology $\mathcal{S} = (T_1, \ldots, T_{i-1}, S_i, T_{i+1}, \ldots, T_k)$.

Now assume that there exists r with $0 \le r \le Ck$ and $A^{\mathcal{S}}(r) < A^{\mathcal{T}}(r)$. Let $X \subseteq$ be given with $|X| = A^{\mathcal{S}}(r)$ and $a^{\mathcal{S}}(X) \ge r$. Since $A^{\mathcal{S}}(r) < a^{\mathcal{T}}(r)$, this implies

$$a^{\mathcal{T}}(X) < r \le a^{\mathcal{S}}(X). \tag{9}$$

For $j \neq i$ we have $a_j^{\mathcal{S}}(X) = a_j^{\mathcal{T}}(X)$, since the trees T_j remain unchanged. For T_i we have to differentiate four cases.

1. If $u, v \notin X$, then $a_i^{\mathcal{S}}(X) = a_i^{\mathcal{T}}(X)$, implying $a^{\mathcal{S}}(X) = a^{\mathcal{T}}(X)$, contradicting (9).

2. If $u, v \in X$, then $a_i^{\mathcal{S}}(X) = a_i^{\mathcal{T}}(X)$, leading to the same contradiction as the preceding case.

3. If $u \in X$ and $v \notin X$, then we have $a_i^{\mathcal{S}}(X) \le a_i^{\mathcal{T}}(X)$, implying $a^{\mathcal{S}}(X) \le a^{\mathcal{T}}(X)$, contradicting (9).

4. If $u \notin X$ and $v \in X$, then set $Y := (X \cup \{u\}) \setminus \{v\}$, ie. is X with v replaced by u. Since v is a leaf in every tree, and u may occur at level 1, the exchange of u and v increases $a_j^{\mathcal{T}}$, leading to

$$a_j^{\mathcal{S}}(Y) = a_j^{\mathcal{T}}(Y) \ge a_j^{\mathcal{T}}(X) = a_j^{\mathcal{S}}(X)$$

for $j \neq i$. At the same time we have

$$a_i^{\mathcal{T}}(Y) = a_i^{\mathcal{S}}(X),$$

and hence

$$a^{\mathcal{T}}(Y) \sum_{j=1}^{k} a_j^{\mathcal{T}}(Y) \ge \sum_{j=1}^{k} a_j^{\mathcal{S}}(X) = a^{\mathcal{S}}(X) \ge r.$$

Therefore $A^{\mathcal{T}}(r) \le |Y| = |X| = A^{\mathcal{S}}(r)$, contradicting the assumption.

Consequently, we have $A^{\mathcal{S}}(r) \ge A^{\mathcal{T}}(r)$ for $0 \le r \le nk$ and hence $\mathcal{S} \succeq \mathcal{T}$. Since the number of clients, which receive no stripe from the server, is one less in $\mathcal{S}$ than in $\mathcal{T}$, without increasing the degree of s, iterated application of this construction leads to a topology $\mathcal{S} \in \mathfrak{D}^{C,k}$ with $\mathcal{S} \succeq \mathcal{T}$.

Since $\mathcal{C}$ is optimal stable in $\mathfrak{D}^{C,k}$, it is optimal stable in $\mathfrak{C}^{C,k}$. □

Theorem 6.4. *$\mathcal{C}$ is optimal stable in $\mathfrak{C}^{C,k}$.*

Proof. For every topology $\mathcal{T} \in \mathfrak{C}^{C,k}$ there exists a topology $\mathcal{S} \in \mathfrak{D}^{C,k}$ with with $\mathcal{S} \succeq \mathcal{T}$, by LEmma 6.3. Since $\mathcal{C}$ is optimal in $\mathfrak{D}^{C,k}$, we have $\mathcal{C} \succeq \mathcal{T}$, due to the fact, that $\succeq$ is transitive. □

A A simple fact about monotone sequences of integers

Lemma A.1. *Let* $(X_i)_{1\le i\le n}$ *be a finite sequence of integers, and* $x := \sum_{i=1}^n x_i$.

(1) If (x_i) *is non-increasing, then*

$$\sum_{i=1}^{j} x_i \ge j \left\lfloor \frac{x}{n} \right\rfloor .$$

(2) If (x_i) *is non-decreasing, then*

$$\sum_{i=1}^{j} x_i \le j \left\lceil \frac{x}{n} \right\rceil .$$

Proof. First we prove (1). Since (x_i) is non-increasing, we have $x_1 \ge x_i$ for $1 \le i \le n$, and hence $x \le nx_1$, implying $x_1 \ge \lceil \frac{x}{n} \rceil \ge \lfloor \frac{x}{n} \rfloor$.

Now assume that there exists $2 \le j \le n$ with

$$\sum_{i=1}^{j} x_i < j \left\lfloor \frac{x}{n} \right\rfloor .$$

Choose j to be minimal among those satisfying the previos inequality. Then

$$\sum_{i=1}^{j-1} x_i \ge (j-1) \left\lfloor \frac{x}{n} \right\rfloor .$$

This implies

$$x_j = \sum_{i=1}^{j} x_i - \sum_{i=1}^{j-1} x_i \le j \left\lfloor \frac{x}{n} \right\rfloor - (j-1) \left\lfloor \frac{x}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor ,$$

and therefore, due to the monotony of (x_i),

$$x_i < \left\lfloor \frac{x}{n} \right\rfloor \quad \text{for } i \ge j.$$

As a consequence, we have

$$\sum_{i=j+1}^{n} x_i < (n-j) \left\lfloor \frac{x}{n} \right\rfloor$$

and hence,

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{j} x_i + \sum_{i=j+1}^{n} x_i < i \left\lfloor \frac{x}{n} \right\rfloor + (n-i) \left\lfloor \frac{x}{n} \right\rfloor \le x,$$

contradicting the definition of x.

Statement (2) is a consequence of (1), since $(y_k)_{1\le i\le n}$ with $y_i := x_n - x_i$ is non-decreasing. □